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OPTIMIZATION, SYSTEM ANALYSIS, AND OPERATIONS RESEARCH

Stability Analysis of “Bridge–Pedestrians” System Based on Tsypkin Criterion

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Abstract—A novel cybernetic model has been developed to analyze the dynamics of the “bridge–pedestrians” system in the transverse direction, incorporating the functional state of the pedestrians. An analytical expression has been derived for the critical number of pedestrians capable of inducing rocking in the bridge. Additionally, the stability region of the system has been assessed using the frequency criterion established by Ya.Z. Tsypkin, specifically applied to the parameters of the London Millennium Bridge. The results of this study indicate that the rocking of the bridge may be attributed to a minor neuromuscular delay among pedestrians, rather than to the synchronization of their steps, as suggested in several existing publications. Furthermore, the obtained results may have broader implications for other classes of oscillatory human–machine systems.

Keywords: stability, reliability of structures, swinging bridge, footbridges, London Millennium Bridge

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1. INTRODUCTION

Over the past two centuries, numerous incidents involving pedestrian bridges have been documented, including the notable swaying of London’s Millennium Bridge [1]. Constructed to commemorate the arrival of the third millennium, the bridge features a lightweight suspension structure characterized by cables positioned below deck level. The Millennium Bridge stands out as one of the few structures for which extensive and valuable observational data has been gathered. Notably, it has been observed that lateral sway increases with the number of pedestrians and diminishes when pedestrian traffic decreases or comes to a halt.

The incident involving the Millennium Bridge prompted a significant surge of publications by esteemed researchers in prestigious scientific journals [2–5]. Initial studies concluded that the large amplitude of oscillations was primarily induced by the synchronous stepping of pedestrians. This finding not only resonated with public perception but also aligned well with the theoretical framework of synchronization in coupled oscillators [6]. However, subsequent observational data emerged that could not be solely explained by synchronization. For instance, oscillations unrelated to the average step frequency were recorded, as well as the identification of a specific critical number of pedestrians capable of inducing rocking in the bridge [1, 7]. In light of this evidence, several researchers proposed that synchronization may be a consequence rather than a causative factor in the rocking of the bridge [4, 5, 8–10].

In this paper, a novel model for the dynamics of the “bridge–pedestrians” system in the transverse direction, incorporating the functional state of pedestrians through a delay link, is proposed. Based on this new model, an innovative approach to analyze the stability of the system is introduced.

Existing approaches to the analysis of the “bridge–pedestrians” system often describe the system model in terms of mechanics and the influence of dynamic forces in both time and frequency domains [11, 12]. The most commonly encountered dynamic model of a bridge in the literature is represented by the following equation [8, 12, 13]

$$M\ddot{x}(t) + C\dot{x}(t) + Kx(t) = F(t), \tag{1}$$

where M, C and K are the mass, damping and stiffness matrices, $\ddot{x}(t)$, $\dot{x}(t)$ and $x(t)$ are the acceleration, velocity and displacement vectors, $F(t)$ is the vector of external forces, which is defined as [14]

$$F(t) = G_p + \sum_{i=1}^{n} G_p \alpha_i \sin(2\pi i f t - \phi_i), \tag{2}$$

where G_p is the human weight, α_i is the Fourier coefficient of the ith harmonic, $f(t)$ is the frequency, ϕ_i is the phase shift of the ith harmonic, i is the ordinal number of the harmonic, and n is the total number of harmonics.

By analogy with (1), the dynamics of pedestrians can be modeled as an oscillator characterized by its own mass, stiffness, and damping coefficient. This methodology has been applied to analyze vertical oscillations in [15, 16], where the “bridge–pedestrians” system is expressed as

$$\begin{bmatrix} m_s & 0 \\ 0 & m_c \end{bmatrix} \begin{Bmatrix} \ddot{x}_s(t) \\ \ddot{x}_c(t) \end{Bmatrix} + \begin{bmatrix} c_s + c_c & -c_c \\ -c_c & c_c \end{bmatrix} \begin{Bmatrix} \dot{x}_s(t) \\ \dot{x}_c(t) \end{Bmatrix} + \begin{bmatrix} k_s + k_c & -k_c \\ -k_c & k_c \end{bmatrix} \begin{Bmatrix} x_s(t) \\ x_c(t) \end{Bmatrix} = \begin{Bmatrix} f_s(t) \\ f_c(t) \end{Bmatrix}. \tag{3}$$

In equation (3), m, c, and k denote the mass, damping coefficient, and stiffness, respectively; the index s corresponds to the bridge, while c pertains to the pedestrian.

The inverted pendulum model with a rigid support and limited motion in the frontal plane effectively captures key features of pedestrian behavior on a horizontally oscillating surface, including both kinematics and kinetics [9, 17]. One of the assumptions underlying this model is that ground surface oscillations do not influence the timing of pedestrian steps. However, as demonstrated in [18], this assumption does not always hold true in practice. In this paper, a law for controlling foot placement that accounts for the delay in foot contact with the ground is proposed.

Recent advancements in addressing the stability of pedestrian bridges are presented in works such as [8–10], where results are derived based on the assumption that step synchronization arises as a consequence of bridge swaying. This condition facilitates the formulation of a relationship between the amplitude and phase balance of pedestrians and the bridge, allowing for the determination of a critical number of pedestrians that satisfy this relationship. In [8–10], pedestrian dynamics are modeled using the Van der Pol oscillator

$$f(x, \dot{x}) = \lambda(\dot{x}^2 + x^2 a^2)\dot{x} + \omega^2 x, \tag{4}$$

where x is the coordinate of the pedestrian’s center of mass, λ is the damping, a is the limit cycle amplitude, ω is the step frequency. In the subsequent work by the authors [8], the force $F(t)$ exerted on the bridge by pedestrians is articulated in terms of the average pedestrian damping coefficient $\overline{\sigma}(\overline{\omega_i}, \Omega)$. This coefficient is significantly influenced by the ratio of the bridge oscillation frequency Ω to the pedestrian step frequency $\overline{\omega_i}$. It was found that there is a large range of pedestrian step frequencies and bridge oscillations for which $\overline{\sigma}(\overline{\omega_i}, \Omega) < 0$. This means that, at a certain critical number of pedestrians, the overall modal damping of the bridge may become negative. Consequently, the authors proposed a straightforward formula for calculating the critical number of pedestrians

$$N_{cr} = -c_0/\overline{\sigma}, \tag{5}$$

where c_0 is the passive damping coefficient of the bridge.

The paper is structured as follows: Section 2 presents the problem statement. Section 3 describes the model of the "bridge–pedestrians" system. Section 4 discusses the stability analysis of this system and provides analytical expressions for the critical number of pedestrians. Finally, Section 5 outlines the results and their potential applications.

2. PROBLEM STATEMENT

The existing literature employs an approach to modeling pedestrian behavior that emphasizes understanding the mechanisms of walking as governed by the central nervous system. However, the high sensitivity of humans to surface vibrations elicits a response that triggers subsequent muscular actions. This observation underscores the necessity of conceptualizing humans as integral components within a closed system, taking into account their physical and psychophysiological properties. This perspective is well-established in the field of human-machine systems and is predicated on the characterization of human functional states [19]; however, it has yet to be applied to the dynamics of gait.

Without delving into the causes of bridge swaying, we will assume that individuals traverse the bridge at an average step frequency, exerting force on the surface equivalent to their body weight while simultaneously striving to maintain balance through visual and vestibular information processed by the central nervous system. The corresponding block diagram for such a system is presented in Fig. 1. In this context, this paper proposes the development of a cybernetic model for the "bridge–pedestrians" system and aims to investigate its applicability for designing and analyzing bridge structural vibrations using methodologies from automatic control theory. To achieve this, it is essential to delineate the dynamics of movement for each component within the "bridge–pedestrians" system through dynamic links.

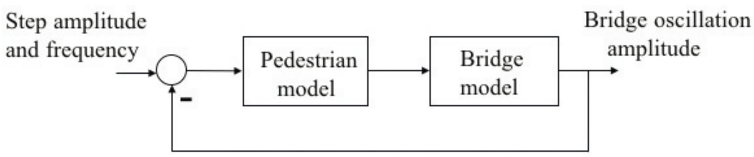

Fig. 1. General diagram of the "bridge–pedestrians" system.

3. MODEL OF THE "BRIDGE–PEDESTRIANS" SYSTEM

Human movement encompasses a variety of types that are often difficult to characterize due to their inherent randomness. Consequently, existing literature has examined the effects of groups of individuals engaged in walking [15], running [20], and jumping [21] on structural design considerations. By defining specific tasks and categorizing human movements, it becomes possible to introduce an approximate mathematical description of these actions that captures the fundamental properties of locomotion. Such descriptions are particularly relevant in the fields of bipedal robot design and human–machine systems.

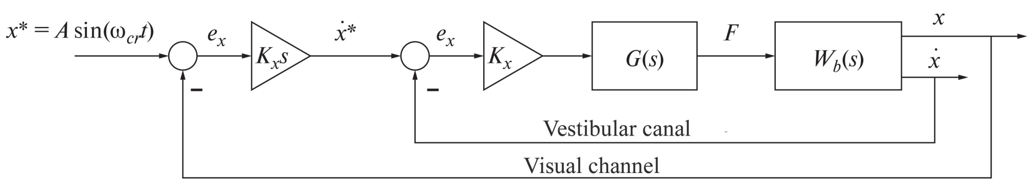

Fig. 2. Block diagram of the "bridge–pedestrians" system.

In this context, the findings from cybernetic models of pilot control actions — specifically concerning error tracking and pitch deviation rates — are well-documented. These models have contributed to our understanding of oscillatory phenomena induced by pilot actions [22, 23]. Research into the interaction between human pilots and aircraft has revealed a tendency for individuals to strive for optimal system control, which manifests as an adaptive property of neuromuscular dynamics in response to the changing dynamics of the system [19, 22–25].

The concept of optimality in human movement is frequently discussed in relation to the energy expenditure associated with executing specific actions. This characteristic is particularly applicable to periodic and repetitive movements, such as sustained walking. On a stable surface, the primary objective is to maintain balance, a task that individuals typically perform reflexively, without conscious thought. However, when navigating an unstable surface, individuals must exert effort or control to sustain balance. This process requires time for the central nervous system to process information and make decisions, which introduces a certain degree of delay.

A clear illustration of the interaction between a person and a structure can be observed in everyday situations, such as when traversing a relatively lightweight suspension bridge on a two-wheeled vehicle (e.g., a bicycle). In such scenarios, the bridge begins to sway noticeably. Notably, the more actively an individual attempts to maintain balance, the more pronounced the oscillations of the bridge become. In all instances, consciously reducing one's efforts to maintain balance — essentially decreasing the proportional gain coefficient "in the head" — can mitigate these oscillations.

The cybernetic model of human behavior in the frequency domain is structured as a series of interconnected blocks, each representing the processes of perception, strategy development, and control action processing [25, 26]. Three primary stimuli facilitate the perception of information: visual, vestibular, and proprioceptive. Within the framework of a structural approach, it is posited that the processes of information processing and strategy formulation occurring within the central nervous system are analogous for each type of perceptual stimulus. Each action performed necessitates a specific duration of time, which can be effectively characterized by a delay link; this delay tends to increase as the complexity of the control process escalates.

The aforementioned adaptive property of humans is represented through correction blocks corresponding to each perceived stimulus, with the cumulative response subsequently directed to the motor system. This structure ultimately delineates the transfer function governing human control actions [25, 27–29].

The most extensively studied model is the correction model developed by individuals based on visual perception of a command stimulus. Numerous studies have demonstrated that individuals possess the ability to amplify, differentiate, and smooth the perceived signal [27–29].

Figure 2 illustrates the model of a pedestrian who utilizes both visual and vestibular channels for information perception while walking. According to this model, a pedestrian attempts to compensate for discrepancies in the angle and angular velocity of roll to maintain balance during locomotion. Thus, the pedestrian operates within a closed-loop system, with their behavior being influenced by the dynamics of the bridge.

The transfer function of the pedestrian model relating the error in angle deviation to the roll angle can be expressed as [24, 30, 31]

$$W_p(s) = NK_x s K_{\dot{x}} G(s), \tag{6}$$

$$G(s) = \frac{T^2 e^{-\tau s}}{s^2 + 2\xi T s + T^2}, \tag{7}$$

where N represent the number of pedestrians, while K_x and $K_{\dot{x}}$ denote the gain factors, $G(s)$ is the transfer function governing the neuromuscular dynamics of pedestrians, ξ and T refer to the damping factor and frequency, respectively, whereas τ is the neuromuscular lag time. It is important to note that, as demonstrated in [31], the value of $K_{\dot{x}}$ is negative.

Considering the bridge model as described in equations (1) and (6), the transfer function of the open-loop system comprising the "bridge–pedestrians" interaction — from the displacement in the transverse direction of the bridge x to the deviation error e_x — can be expressed as follows

$$W(s) = W_p(s) W_b(s) = \frac{NK_p T^2 s e^{-\tau s}}{(s^2 + 2\xi T s + T^2)(Ms^2 + Cs + K)}, \tag{8}$$

where $K_p = K_x K_{\dot{x}}$.

4. STABILITY ANALYSIS OF THE "BRIDGE–PEDESTRIANS" SYSTEM

The remaining parameters of the system are assumed to be constant. The following parameters of the London Millennium Bridge are known: mass $M = 81\,000$, stiffness $K = 3\,390\,733$ kg/s^2, damping coefficient $C = 7681$ kg/s, and natural frequency $\Omega = \sqrt{K/M} = 6.5$ rad/s [8, 33]. Additionally, the parameters related to the neuromuscular dynamics of a pedestrian are $T = 30$, $\xi = 0.7$ [31].

The comfortable time required for information processing in the central nervous system and the transmission of signals along neuromuscular fibers for pilots in manual control mode is approximately 0.2 seconds [25, 27]. Due to the movement of the bridge surface, the pedestrian's orientation angle changes, presenting a non-standard situation that triggers the adaptation process to new conditions. This adaptation is reflected in the settings of the parameters defined in equation (6), including the delay time. Depending on external circumstances, an individual may either decrease or increase the neuromuscular delay time. For instance, a reduction in the delay to 0.08 seconds is associated with an increase in neuromuscular tension [22].

To evaluate the magnitude of the delay time and its corresponding frequency, which affect the stability of the system, it is convenient to employ the frequency criterion established by Ya.Z. Tsypkin [34, 35]. The critical frequency ω_{0i} is determined from the following equation

$$|W(\omega_{0i}, N)| - 1 = 0, \tag{9}$$

where $|W|$ denote the amplitude-frequency response (AFR) of an open-loop system without the delay link, as presented in (8). Subsequently, the resulting value of ω_{0i} is substituted into the expression for the phase relationship, which generally assumes the following form

$$\tau_{0i}(n) = \frac{\theta(\omega_{0i})}{\omega_{0i}} + \frac{2\pi n}{\omega_{0i}}, \quad n \in \mathbb{N}, \tag{10}$$

where $\theta(\omega_{0i}) = \arctan(W(\omega_{0i}))$. This critical delay time τ_{0i} delineates the transition of the system's roots through the imaginary axis, thereby establishing the stability boundary of the system. The system under investigation will be considered stable when (9) yields has no solutions with respect to ω_{0i}; in other words, when the hodograph of the system remains within the unit circle.

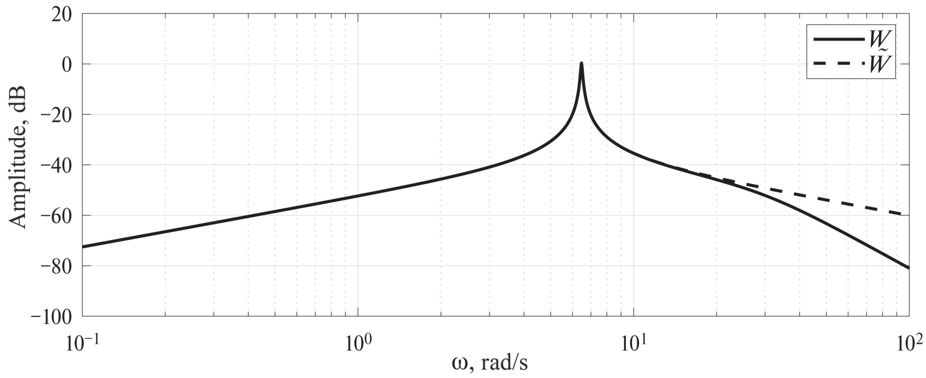

Fig. 3. Amplitude-frequency response of the “bridge–pedestrians” system with and without the neuromuscular dynamics link.

Let us apply the criteria outlined in (9) and (10) to (8). It is noteworthy that the AFR (8) coincides with the AFR that does not account for the neuromuscular dynamics link across a broad frequency range of 1–10 rad/s (see Fig. 3). This observation allows us to infer that for the purpose of assessing the stability of the system, the neuromuscular dynamics link can be neglected within this frequency range. Consequently, equation (8) can be reformulated as follows

$$\tilde{W}(s) = \frac{NK_p s e^{-\tau s}}{(Ms^2 + Cs + K)}. \tag{11}$$

Let us move from s to $j\omega$ in (8) and isolate the real and imaginary components without taking into account the delay link

$$\tilde{W}(j\omega) = K_p N\left[\frac{j\omega(K - M\omega^2)}{(K - M\omega^2)^2 + C^2\omega^2} + \frac{C\omega^2}{(K - M\omega^2)^2 + C^2\omega^2}\right]. \tag{12}$$

Then (9) can be written as

$$\frac{K_p^2 N^2\Big[(C\omega^2)^2 - \omega^2(K - M\omega^2)^2\Big]}{\Big[(K - M\omega^2)^2 + C^2\omega^2\Big]^2} - 1 = 0, \tag{13}$$

by expanding the brackets, we derive an eighth-order equation represented as

$$A_8\omega^8 + A_6\omega^6 + A_4\omega^4 + A_2\omega^2 + A_0 = 0, \tag{14}$$

where $A_8 = -M^4$, $A_6 = 4KM^3 - K_p^2 M^2 N^2 - 2C^2M^2$, $A_4 = 2KK_p^2 MN^2 - 6K^2M^2 + C^2K_p^2N^2 + 4C^2KM$, $A_2 = 4K^3M - K^2K_p^2N^2 - 2C^2K^2$, $A_0 = -K^4$.

We perform a variable substitution in equation (14) by letting $\omega^2 = t$. This substitution is essential for the subsequent analysis of the obtained solution. The solution to equation (14) with respect to ω, expressed symbolically using MATLAB, yields the following expression

$$\begin{aligned} t^2 = \Bigg[&\frac{2}{4M^2}\sqrt{C^4 + \frac{\sigma_7}{2} + 4KM^3\sigma_3 + 3C^2\mathrm{K_p}^2N^2 - 2C^2M^2\sigma_3 - \sigma_2 - \mathrm{K_p}^2M^2N^2\sigma_3 - \sigma_1} - 2C^2 \\ &+ 2M^2\sqrt{\frac{\sigma_4^2}{4M^8} + \frac{\sigma_6 - 8MK^3 + \sigma_5}{\sigma_4} + \frac{-C^4 + \sigma_2 + C^2\mathrm{K_p}^2N^2 - 6K^2M^2 + \sigma_1}{M^4}} - \mathrm{K_p}^2N^2 + 4KM\Bigg]^{\frac{1}{2}}, \end{aligned} \tag{15}$$

where $\sigma_1 = 2K{\rm K_p}^2MN^2$, $\sigma_2 = 4C^2KM$, $\sigma_3 = \sqrt{\frac{\sigma_7}{4M^4} - \frac{2K^2}{M^2} - \frac{8K^3M}{\sigma_4} + \frac{\sigma_6}{\sigma_4} + \frac{2C^2{\rm K_p}^2N^2}{M^4} + \frac{\sigma_5}{\sigma_4}}$,
$\sigma_4 = 2C^2M^2 + {\rm K_p}^2M^2N^2 - 4KM^3$, $\sigma_5 = 2K^2{\rm K_p}^2N^2$, $\sigma_6 = 4C^2K^2$, $\sigma_7 = {\rm K_p}^4N^4$.

The expression in equation (15) appears under the square root and is dependent on the variable parameter representing the number of pedestrians N. Consequently, numerically, equation (15) can assume any values, including complex ones. However, the physical context of the problem necessitates that only real quantities are considered. Therefore, it is imperative to establish a condition for the existence of a real non-negative solution. One such condition is the non-negativity of the radical expression in equation (15). Based on this, all conditions for a real solution are derived symbolically using MATLAB

$$\begin{aligned} N \in \mathbb{R} \wedge 2\,M^2\sqrt{\frac{{\sigma_5}^2}{4\,M^8} + \frac{\sigma_7 - 8\,M\,K^3 + \sigma_6}{\sigma_5} + \frac{-C^4 + \sigma_3 + C^2\,{\rm Kp}^2\,N^2 - 6\,K^2\,M^2 + \sigma_1}{M^4}} \\ + 2\sqrt{C^4 + \frac{\sigma_8}{2} + 4\,K\,M^3\,\sigma_4 - 2\,C^2\,M^2\,\sigma_4 + 3\,C^2\,{\rm Kp}^2\,N^2 - \sigma_3 - {\rm Kp}^2\,M^2\,N^2\,\sigma_4 - \sigma_1 - \sigma_2} \\ = 2\,C^2 - 4\,K\,M \wedge 2\,C^2 + \sigma_2 \neq 4\,K\,M \wedge 0 < N, \end{aligned} \tag{16}$$

where $\sigma_1 = 2\,K\,{\rm Kp}^2\,M\,N^2$, $\sigma_2 = {\rm Kp}^2\,N^2$, $\sigma_3 = 4\,C^2\,K\,M$,
$\sigma_4 = \sqrt{\frac{\sigma_8}{4\,M^4} - \frac{2\,K^2}{M^2} - \frac{8\,K^3\,M}{\sigma_5} + \frac{\sigma_7}{\sigma_5} + \frac{2\,C^2\,{\rm Kp}^2\,N^2}{M^4} + \frac{\sigma_6}{\sigma_5}}$, $\sigma_5 = 2\,C^2\,M^2 + {\rm Kp}^2\,M^2\,N^2 - 4\,K\,M^3$,
$\sigma_6 = 2\,K^2\,{\rm Kp}^2\,N^2$, $\sigma_7 = 4\,C^2\,K^2$, $\sigma_8 = {\rm Kp}^4\,N^4$.

After conducting both numerical and analytical analyses of the aforementioned constraints in MATLAB, we determine that the smallest N for which a real solution exists is governed by the condition

$$K_p^4N^4 + (4MKK_p^2 - 2C^2K_p^2)N^2 + C^4 - 4MKC^2 \geqslant 0. \tag{17}$$

By setting the left side of equation (17) to zero and substituting $N^2 = t_2$, we obtain an expression for the discriminant

$$D_2 = (4MKK_p^2 - 2C^2K_p^2)^2 - 4K_p^4(C^4 - 4MKC^2) = 16K_p^4M^2K^2, \tag{18}$$

$$\sqrt{D_2} = \pm 4K_p^2MK, \tag{19}$$

then the roots of (17) can be found from the expression

$$t_2^1 = \frac{-4MKK_p^2 + 2C^2K_p^2 + 4K_p^2MK}{2K_p^4}, \tag{20}$$

$$t_2^2 = \frac{-4MKK_p^2 + 2C^2K_p^2 - 4K_p^2MK}{2K_p^4}. \tag{21}$$

Substituting the numerical parameters into equations (20) and (17), we find that $t_2^1 > 0$ and $t_2^2 < 0$. We will subsequently perform the inverse substitution of t_2 back to N^2 and extract the root of t_2. Therefore, we will discard $t_2^2 < 0$, leading to

$$N^2 = \frac{-4MKK_p^2 + 2C^2K_p^2 + 4K_p^2MK}{2K_p^4}, \tag{22}$$

from where, leaving only the positive root, we get:

$$N = \frac{C}{K_p}. \tag{23}$$

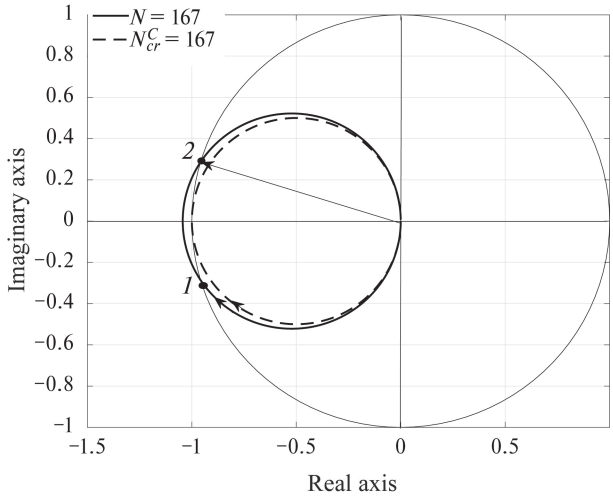

Fig. 4. Amplitude-phase frequency response of an open-loop system with varying numbers of pedestrians.

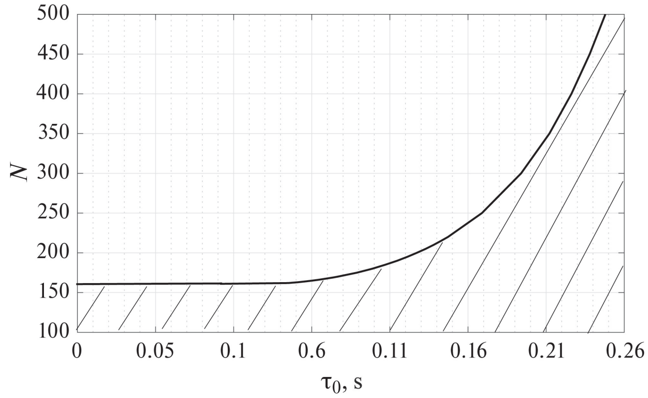

Fig. 5. The dependence of the number of pedestrians on the critical neuromuscular delay is indicated. The region of stability is marked with hatching.

By substituting the system parameters into equation (23), we obtain $N = 160.0208$, which corresponds to the critical number of pedestrians derived from Ya.Z. Tsypkin's criterion, denoted as $N_{cr}^C = 160$. Thus, this value represents the maximum possible number of pedestrians at which the system remains stable, independent of delay. The graphical representation of solution (9) is illustrated in Fig. 4, where the numerical result coincides with that obtained from equation (23).

Further increases in N lead to the intersection of the hodograph with the unit circle at two points. For instance, at $N = 167$, we find two solutions: $\omega_{01} = 6.45$ rad/s and $\omega_{02} = 6.48$ rad/s, with the corresponding points indicated in Fig. 4. Since $\omega_{02} > \omega_{01}$, it follows that $\tau_{02} < \tau_{01}$, where τ_{02} represents the critical delay time. For the given hodograph, this can be determined from the expression

$$\tau_{02} = \frac{\pi - \theta(\omega_{02})}{\omega_{02}}. \tag{24}$$

Substituting the numerical values into equation (24), we obtain $\tau_{02} = 0.086$ s. Thus, for $N = 167$, the critical delay for stability is $\tau_{02} = 0.086$ s, which corresponds to excessive neuromuscular tension in individuals [22]. An increase in the number of pedestrians further leads to an escalation in the critical delay required for maintaining stability.

Figure 5 illustrates the relationship between the number of pedestrians and the neuromuscular delay $N(\tau_0)$, as derived from the solutions of equations (9) and (24). The figure clearly indicates

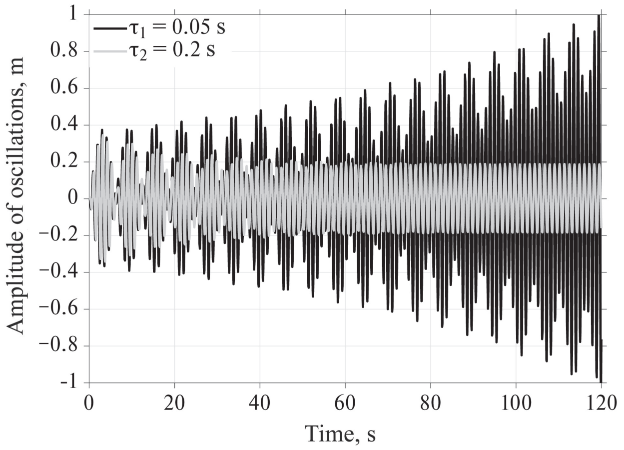

Fig. 6. Dependence of the amplitude of bridge oscillations on time for 250 pedestrians walking at a frequency of 5.4 rad/s.

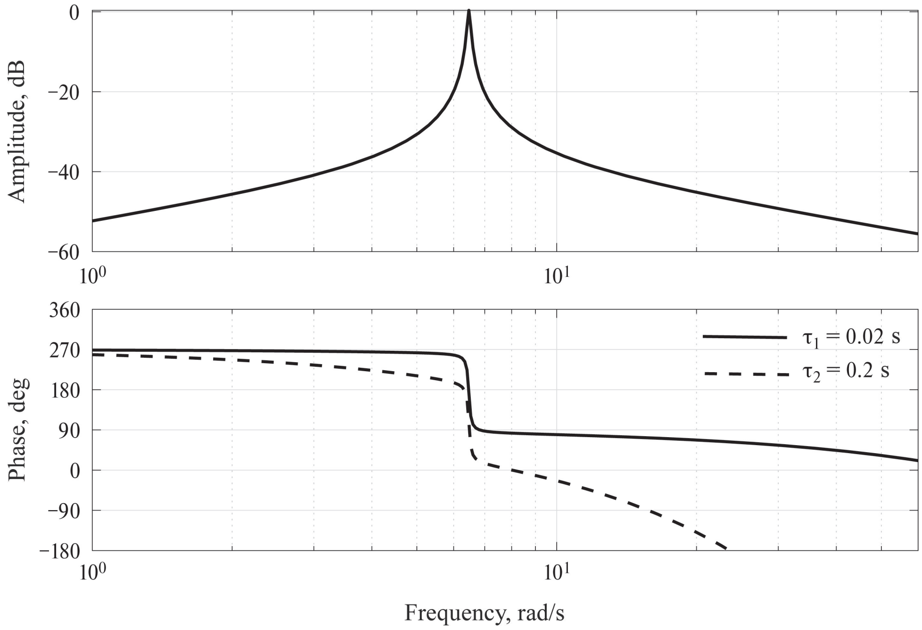

Fig. 7. Frequency response of the "bridge–pedestrians" system with different values of neuromuscular delay.

that the stability region is constrained by the number of pedestrians at minimal delays, with a sharp increase observed in the region of delay values that are typical for humans.

The dependence of the bridge oscillation amplitude on time, influenced by 250 pedestrians walking at an average frequency of 5.4 rad/s and exhibiting varying delays, is depicted in Fig. 6. At a normal delay of $\tau_2 = 0.2$ s, the bridge exhibits stable oscillations with an amplitude of approximately 20 cm. In contrast, the neuromuscular tension associated with a smaller delay of $\tau_1 = 0.05$ s results in a gradual increase in oscillation amplitude.

Figure 7 presents the amplitude-phase frequency response (APFR) of the "bridge–pedestrians" system for different values of neuromuscular delay (0.02 s and 0.2 s), which introduces a corresponding phase shift between the input and output of the system.

Thus, the stability of the system is influenced not only by the number of pedestrians but also by the delays they introduce due to the characteristics of their neuromuscular systems. This factor had not been previously considered in the analysis of the "bridge–pedestrians" system.

5. CONCLUSION

This paper proposes a novel approach to studying the stability of human-machine systems with oscillatory dynamics, exemplified by the “bridge–pedestrians” system. A linear model of the closed-loop “bridge–pedestrians” system is examined, incorporating both pedestrian muscle dynamics and processes occurring within the central nervous system. This approach enables us to represent the closed-loop system as a transfer function that includes a delay element, thereby facilitating stability assessment using methods from automatic control theory.

The applicability of this approach is demonstrated through an analysis of the swinging of the London Millennium Bridge during its opening day, when pedestrians traversed it. Utilizing Ya.Z. Tsypkin’s frequency criterion across various delay values, we derived conditions pertaining to the number of pedestrians that ensure the stability of the closed-loop system is not compromised. Numerical results indicate that significant bridge oscillations can be attributed to reduced neuromuscular delays among pedestrians. Specifically, due to their heightened sensitivity to minor surface vibrations, pedestrians require time to adapt to changing conditions, which manifests as excessive neuromuscular tension. A rapid pedestrian response introduces a minimal phase shift between the input and output of the “bridge–pedestrians” system, resulting in bridge swaying and potential loss of stability. Conversely, a slower response introduces a phase shift of approximately 90 degrees, which contributes to stabilization.

The proposed approach provides a more nuanced framework for analyzing and designing structures utilized by people that exhibit oscillatory dynamics. Future research could focus on refining the parameters of the neuromuscular dynamics model while considering nonlinearities in the bridge model and the impact of external disturbances.

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